CHAPTER 11

MINIMIZING ENGINEERING EFFORT

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NOMENCLATURE

a	Distance, range number, bilaterial tolerance
b	Width, range number
C	Constant
D	Helix diameter
dim	Dimensional operator
E	Young's modulus
E_n	Error using n applications of Simpson's rule
e_i	The ith exponent
f	Function
$f^{(i)}$	The ith derivative of function f
F	Fundamental dimension of force, fractional reduction of interval of uncertainty
g	Function
h	Function, ordinate spacing
i	Index
I	Second area moment, value of integral
I_i	Approximate value of integral using i applications of Simpson's rule
k	Spring rate

K_{ij} Exponent of fundamental dimension in row i, of parameter j in dimensional matrix

L Fundamental dimension of length

ℓ Span, left

ln Natural logarithm

m Mass, subscript of model

n Number

N Number of experiments to establish a robust functional relationship among n parameters, number of function evaluations

N' Number of experiments to establish a robust functional relationship among dimensionless parameters

N_a Number of active turns in a spring

N_π Number of pi terms in a complete set

p Number of points necessary to establish a robust functional relationship between two parameters

P Load

Q Fundamental dimension of charge

r Rank of dimensional matrix, right

s Scale factor, the ratio of model over prototype dimension

T Fundamental dimension of time

x Location parameter

x^* Abscissa of extreme of a function

x_ℓ, x_r Range numbers on left and right, respectively

y Transverse beam deflection

Δ Tolerable error

θ Fundamental dimension of temperature

π_i The ith pi term

ξ Location in Simpson's rule application interval where error term is exact

11.1 INTRODUCTION

The old carpenter's admonition "Measure twice, cut once" reminds us that sound preparatory effort avoids later grief in terms of redoing or scrapping prior work effort. In technical undertakings, engineering effort is required long before work starts. Not only must it be done correctly, but since it is an overhead cost, it is important that it be accomplished in a cost-efficient manner without compromising the quality of the result. In order to accomplish this routinely, engineers have developed and adopted strategies, manners of approach that are routinely mindful of effective use of engineering resources.

One such strategy is the mathematical model. It gives us quantitative insight into domains that are new to us. It is unfortunate that the name *mathematical model* is commonly applied to this tool, for mathematics does not intrinsically contain the reality. It has to be carefully built in if the model is to satisfactorily describe nature.

Attention focus for thinking and communicative processes is rooted in and well served by concepts of system, boundary, and surroundings or control region, control surface, and surroundings. There are also notions of cause, effect, and extent as systems interact with their surroundings. We recognize heat and work effects, tractive effects, charge effects, chemical effects, and ballistic effects related to nuclear phenomena. It is in these effects (and their quantitative expression) that reality is modeled. It is when these effects are combined with notions of accountability, or balances, and first principles that reality can be incorporated into mathematical models ([11.1], Chaps. 6, 7).

Deterministic, deductive mathematical models are usually created using the following steps ([11.1], p. 228):

1. *Isolate* a finite or infinitesimal system or control region.
2. *Identify* the significant influences of the surroundings, or changes within the isolated system or control region.
3. *Qualify* significant influences or changes with mathematical models of effects.
4. *Relate* influences to system or control-region behavior by using first principles.
5. *Limit,* if necessary, as Δx, Δy, Δz, Δt, etc., approach zero.
6. *Solve* the resulting equation(s) for variable(s) of interest. Assumptions or judgments may be required to make a solution possible.
7. *Check* your work (see Sec. 11.6).

Engineers recognize that variability is omnipresent in nature and that measured quantities are knowable only in terms of estimates of means and variances, distributional forms, and confidence limits. This variability or uncertainty must be considered when judging the worth of the model results.

11.2 REDUCING THE NUMBER OF EXPERIMENTS

In describing the functional relationship between variables x_1 and x_2, it takes a number of experiments (points) to establish a satisfactory approximation to the functional relationship. Consider that number of experiments to be p. At this point we are concerned not with the method of establishing the working approximation (least-square curve fits, for example) but with the amount of effort associated with gathering the data points used to establish that relationship. If the level of effort in time and expense is proportional to the number of points p, we use the magnitude of p as our index to cost. The relationship between x_1 and x_2 can be displayed as a data string on a sheet of graph paper ([11.1], pp. 139–160).

How many experiments are necessary to describe a phenomenon involving n parameters $x_1, x_2, \ldots, x_n$? During the experiments necessary to relate x_1 to x_2, all other parameters were held constant. The role of x_3 is then introduced by performing p experiments at level $(x_3)_1, (x_3)_2, \ldots, (x_3)_p$. This places p contours on the x_1x_2 graph. Up to this point there have been p^2 experiments. The introduction of the third parameter increased the level of effort *exponentially.* Similarly, the fourth parameter requires p pages of p curves of p points each. The total number of experiments N necessary for n parameters using p points for each curve is, therefore,

$$N = p^{n-1} \tag{11.1}$$

If $p = 6$ and $n = 5$, then $N = 6^{5-1} = 1296$ experiments. If the cost of experimental determination is \$100 or \$1000 per point, then quantitative understanding is prohibitively expensive. Is there any alternative to this investment of time and effort?

We are indebted to Buckingham, who suggested clustering parameters in dimensionless groups. Instead of finding the relationship among

$$f(x_1, x_2, \ldots, x_n) = 0$$

Buckingham suggested finding the relationship among

$$g(\pi_1, \pi_2, \ldots, \pi_{n-r}) = 0$$

where r is the rank of the matrix of dimensions. The level of effort N' is now given by, after Eq. (11.1),

$$N' = p^{n-r-1} \tag{11.2}$$

The ratio N'/N is, using Eqs. (11.1) and (11.2),

$$\frac{N'}{N} = \frac{p^{n-r-1}}{p^{n-1}} = \frac{1}{p^r} \tag{11.3}$$

If the rank of the matrix of dimensions is 2 and 10 points are necessary, then

$$\frac{N'}{N} = \frac{1}{10^2} = \frac{1}{100}$$

and the level of effort has been reduced by a factor of 100.

Pi terms are multiplicative clusters of parameters, formed by exploiting the rule of dimensional homogeneity. The set of *fundamental dimensions* consists of the irreducible set of force F, length L, time T, temperature θ, and charge Q. Mass can be used instead of force. A velocity V has the dimensions of length/time, or L/T, and such quantities are called *secondary* or *derived* quantities. We can say that the dimensions of V, $\dim(V)$, are L/T or L^1T^{-1}, or, more completely,

$$\dim(V) = F^0L^1T^{-1}\theta^0Q^0 \tag{11.4}$$

Care has to be taken to establish a *complete set* of dimensionless clusters, or pi terms. A complete set means that the pi-term set is the exact counterpart of the parameter set. The first step is to construct a *matrix of dimensions* for the parameter set. If the parameters are $x_1, x_2, \ldots, x_n$ and the fundamental dimensions involved are force F and length L, then the matrix of dimensions is displayed as

	x_1	x_2	...	x_n
F	K_{11}	K_{12}	...	K_{1n}
L	K_{21}	K_{22}	...	K_{2n}

For example, for a helical compression spring, the spring rate k is affected by the number of active turns N_a, wire diameter d, torsional modulus G, and helix diameter D. The dimensions are

$$\dim(k) = F^1L^{-1} \qquad \dim(G) = F^1L^{-2}$$

$$\dim(N_a) = F^0L^0 \qquad \dim(D) = F^0L^1$$

$$\dim(d) = F^0L^1$$

The matrix of dimensions for the spring consists of the display of the exponents of the fundamental dimensions in each of the parameters:

	k	N_a	d	G	D
F	1	0	0	1	0
L	−1	0	1	−2	1

The *rank* of this matrix is the order of the largest nonzero determinant that can be found in the matrix. Since the right-hand determinant

$$\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

is nonzero, the rank r is 2. There may be several of these depending on the sequencing of parameters across the top. It is important for completeness that a nonzero determinant be placed on the right in the matrix of dimensions. The number of multiplicative dimensionless clusters or pi terms N_π is given by

$$N_\pi = n - r \tag{11.5}$$

A pi term is formed by writing

$$\pi_i = k^{e_1}N_a^{e_2}d^{e_3}G^{e_4}D^{e_5} \tag{11.6}$$

The dimensional operator is applied as follows:

$$\begin{aligned}\dim(\pi_i) &= \dim(k^{e_1})\dim(N_a^{e_2})\dim(d^{e_3})\dim(G^{e_4})\dim(D^{e_5}) \\ &= (F^1L^{-1})^{e_1}(F^0L^0)^{e_2}(F^0L^1)^{e_3}(F^1L^{-2})^{e_4}(F^0L^1)^{e_5}\end{aligned}$$

For the force dimension,

$$F^0 = F^{e_1}F^0F^0F^{e_4}F^0$$

The exponent of F must be the same on both sides:

$$0 = (1)e_1 + (0)e_2 + (0)e_3 + (1)e_4 + (0)e_5$$

Note that the coefficients of the exponential equation agree with the first row of the dimensional matrix. In other words, the exponential equation associated with any fundamental dimension can be written *by inspection* from the matrix of dimensions. The two exponential equations are

$$e_1 + e_4 = 0 \qquad \text{(for force dimension)} \tag{11.7}$$

$$-e_1 + e_3 - 2e_4 + e_5 = 0 \qquad \text{(for length dimension)} \tag{11.8}$$

There are two exponential equations (r is 2) and five exponents (n is 5), and so three exponents are mathematically arbitrary. We will choose them so that the first three parameters k, N_a, and d each appear in only one pi term. Such parameters are used to control their pi terms independently, if necessary.

It is useful to display a *matrix of solutions*. There are $n - r = 5 - 2 = 3$ pi terms.

	(k)	(N_a)	(d)	(G)	(D)
	e_1	e_2	e_3	e_4	e_5
π_1	1	0	0		
π_2	0	1	0		
π_3	0	0	1		

Solving Eqs. (11.7) and (11.8) to complete the matrix of solutions is done as follows:

$$e_4 = -e_1$$

$$e_5 = 2e_4 + e_1 - e_3$$

For $e_1 = 1, e_2 = 0, e_3 = 0$, $e_4 = -1$, $e_5 = -2 + 1 = -1$

For $e_1 = 0, e_2 = 1, e_3 = 0$, $e_4 = 0$, $e_5 = 0$

For $e_1 = 0, e_2 = 0, e_3 = 1$, $e_4 = 0$, $e_5 = -1$

The completed matrix of solutions is

	(k)	(N_a)	(d)	(G)	(D)	
	e_1	e_2	e_3	e_4	e_5	
π_1	1	0	0	−1	−1	$\Rightarrow \pi_1 = k^1 G^{-1} D^{-1}$
π_2	0	1	0	0	0	$\Rightarrow \pi_2 = N_a^1$
π_3	0	0	1	0	−1	$\Rightarrow \pi_3 = d^1 D^{-1}$

and the pi terms can be displayed as

$$\pi_1 = \frac{k}{GD} \qquad \pi_2 = N_a \qquad \pi_3 = \frac{d}{D}$$

Recall that if $p = 10$, then the number of experiments from Eq. (11.1) is $N = p^{n-1} = 10^{5-1} = 10{,}000$. By using Buckingham's multiplicative dimensionless clusters, Eq. (11.2) gives $N' = 10^{5-2-1} = 100$.

Can we reduce the hundred experiments even more? If we can introduce information we already know, we can. Two identical springs in series (end to end) have twice the turns and half the spring rate; in other words, $\pi_1\pi_2 = C_1(\pi_3)$. The problem reduces to finding

$$\pi_1\pi_2 = h(\pi_3)$$

Now there are only 10 experiments to be performed. As an aid to partitioning our thinking so that we can deal with one thing at a time, we can use the method of derivatives. Since there are three pi terms in the spring problem, we seek the function

$$\pi_1 = h_1(\pi_2, \pi_3)$$

It follows then that

$$d\pi_1 = \frac{\partial \pi_1}{\partial \pi_2}\, d\pi_2 + \frac{\partial \pi_1}{\partial \pi_3}\, d\pi_3 \tag{11.9}$$

In noting the inverse proportionality between π_1 and π_2 from before, we write

$$\pi_1 = \frac{C_1}{\pi_2} \qquad \frac{\partial \pi_1}{\partial \pi_2} = -\frac{C_1}{\pi_2^2} = -\frac{\pi_1 \pi_2}{\pi_2^2} = -\frac{\pi_1}{\pi_2} \qquad \text{(from prior experience)}$$

When we conduct the p experiments and find $\pi_1/\pi_3^4 = C_2(\pi_2)$ at constant π_2, we have

$$\pi_1 = C_2\, \pi_3^4 \qquad \frac{\partial \pi_1}{\partial \pi_3} = 4C_2\pi_3^3 = 4\,\frac{\pi_1}{\pi_3^4}\,\pi_3^3 = 4\,\frac{\pi_1}{\pi_3} \qquad \text{(from test)}$$

Thus Eq. (11.9) becomes

$$d\pi_1 = -\frac{\pi_1\, d\pi_2}{\pi_2} + 4\,\frac{\pi_1}{\pi_3}\, d\pi_3$$

Dividing through by π_1 renders the equation exact and integrable term by term:

$$\frac{d\pi_1}{\pi_1} = -\frac{d\pi_2}{\pi_2} + 4\,\frac{d\pi_3}{\pi_3}$$

$$\ln \pi_1 = -\ln \pi_2 + \ln \pi_3^4 + \ln C$$

or

$$\pi_1 = C\frac{\pi_3^4}{\pi_2} \tag{11.10}$$

The constant C can be found from the p experiments. Equation (11.10) can be written as

$$k = \frac{d^4 G}{8D^3 N_a}$$

Do not underestimate the power of Buckingham's suggestion *and* the incorporation of a priori knowledge with test results to enormously reduce the effort.

11.3 SIMILITUDE

The first similitude equation of which we have a record dates to the fourth century B.C., when it was recorded by Philon of Byzantium for the ballista [11.2]. It related

what we now call the mass of the projectile to be thrown to the diameter of the torsional springs used as

$$\frac{d_1}{d_2} = \left(\frac{m_1}{m_2}\right)^{1/3} \tag{11.11}^{\dagger}$$

Ever since, engineers have embroidered on this idea with useful results. In the context of Sec. 11.2, this is a relationship between two pi terms. The idea that will be useful to us can be related to the helical spring example of Sec. 11.2. With a spring in hand, one can quantitatively express $\pi_1 = k/GD$. However, knowing that π_1 is 0.5×10^{-5} will not identify the spring parameters. What constructing the pi term has done is map all springs with $\pi_1 = 0.5 \times 10^{-5}$ onto a single coordinate. This suggests that one can *model* one spring with $\pi_1 = 0.5 \times 10^{-5}$ with another that also has $\pi_1 = 0.5 \times 10^{-5}$, but is of differing material, spring rate, and helix diameter. This can be useful in adjusting to size and capacity constraints on test instrumentation.

For a timber beam of cross section b wide and d deep, with a concentrated load P located a distance a from the left support, and a span of ℓ, the transverse deflection y at a distance x from the left support is described by

$$f(y, a, b, d, x, P, E, \ell) = 0$$

or equally as well by Buckingham's pi terms as

$$g\left(\frac{y}{\ell}, \frac{a}{\ell}, \frac{b}{\ell}, \frac{d}{\ell}, \frac{x}{\ell}, \frac{P}{E\ell^2}\right) = 0$$

Suppose we wish to model the timber beam in a different size and material. The function g in model terms is written

$$g\left(\frac{y_m}{\ell_m}, \frac{a_m}{\ell_m}, \frac{b_m}{\ell_m}, \frac{d_m}{\ell_m}, \frac{x_m}{\ell_m}, \frac{P_m}{E_m \ell_m^{\,2}}\right) = 0$$

In order for this to be a model, corresponding pi terms must be identical. Since $y_m/\ell_m = y/\ell$, it follows that

$$y_m = \frac{\ell_m}{\ell} y = sy$$

where s is the *scale factor*, $s = \ell_m/\ell$. The other linear dimensions are

$$a_m = sa \qquad b_m = sb \qquad d_m = sd \qquad x_m = sx$$

The sixth pi terms are equated, from which

$$P_m = \frac{PE_m\ell_m^{\,2}}{E\ell^2} = s^2 \frac{E_m}{E} P$$

The load P_m is the mandatory load on the model corresponding to P. The location at which to measure the transverse deflection is $x_m = sx$. If a steel model is 1/10 size and the prototype load is 4800 lbf, the model load P_m is

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† In a book addressing machine design, shouldn't this be Eq. (1.1)?

$$P_m = 0.1^2 \frac{30 \times 10^6}{1.5 \times 10^6} 4800 = 960 \text{ lbf}$$

and the prototype deflection is $y = y_m/s$.

11.4 OPTIMALITY

The subject of optimality is extensive [11.3], [11.4]. Our purpose here is to examine the efficiency of an optimization process itself, for any internal wasted effort in a computer-coded algorithm is incessantly repeated. A unimodal function is one that monotonically increases, monotonically decreases, or monotonically increases then decreases. If the original interval containing a maximum has the range numbers x_ℓ, x_r and there are n ordinates equally spaced within the interval (but no ordinates at x_ℓ or x_r), then the ordinate spacing is

$$h = \frac{x_r - x_\ell}{n+1}$$

By examining the ordinates, the final interval of uncertainty is reduced to $2h$, and the fractional reduction in the interval of uncertainty is

$$F = \frac{2h}{x_r - x_\ell} = \frac{2(x_r - x_\ell)/(n+1)}{x_r - x_\ell} = \frac{2}{n+1}$$

Solving for n gives, for fractional reduction F and bilateral tolerance, $x^* \pm a$ locations of the extreme, respectively:

$$n = \left[\frac{2}{F} - 1\right]_+ = \left[\frac{x_r - x_\ell}{a} - 1\right]_+ \tag{11.12}$$

When n is not an integer, it is rounded up. For $F = 0.001$,

$$n = \left[\frac{2}{0.001} - 1\right]_+ = [2000 - 1]_+ = 1999$$

Thus, 1999 function evaluations are required. See Ref. [11.1], pp. 278–290.

Instead of expending all ordinates simultaneously, one can spend a few, reduce the interval somewhat, and keep repeating the process. For equally spaced ordinates, the optimal procedure ([11.3], p. 282) is spending n as $3 + 2 + 2 + \cdots$. This is called *interval halving*. The total number of function evaluations N spent this way is

$$N = \left[1 + \frac{2 \ln 1/F}{\ln 2}\right]_{\text{odd}+} = \left[2.88 \ln \frac{x_r - x_\ell}{a} - 1\right]_{\text{odd}+} \tag{11.13}$$

For $F = 0.001$,

$$N = \left[1 + \frac{2 \ln 1/1000}{\ln 2}\right]_{\text{odd}+} = [20.93]_{\text{odd}+} = 21$$

This is a remarkable reduction in effort. One can do better by relaxing the equal spacing stipulation and spending ([11.1], pp. 284–289) ordinates $2 + 1 + 1 + \cdots$. Under these circumstances, for fractional and bilateral tolerance reductions, respectively,

$$N = \left[1 + \frac{\ln F}{\ln 0.618\,033\,989}\right]_+ = \left[2.08 \ln \frac{x_r - x_\ell}{a}\right]_+ \tag{11.14}$$

and the method is called *golden section.* For $F = 0.001$,

$$N = \left[1 + \frac{\ln 0.001}{\ln 0.618\,033\,989}\right]_+ = [15.35]_+ = 16$$

which is approximately three-fourths as many function evaluations as were required for interval halving. Can one do better? The answer is a qualified yes. A Fibonacci search will reduce effort by about one function evaluation at the $F = 0.001$ level, but it is not amenable to predicting the number of function evaluations in advance.

While interval halving may be easier to apply manually, golden section should be coded for the computer. Golden section is used for real root finding of $f(x)$ by maximizing $-|f(x)|$. The root is at the zero-ordinate cusp. Figure 11.1 is the documentation sheet for a golden section subroutine named GOLD. This subroutine has served thousands of users over several decades at Iowa State University and elsewhere. The Fortran coding follows.

```
      SUBROUTINE GOLD(K,XA,XB,F,MERIT1,YBIG,XBIG,XL1,XR1,N)
C     IOWA CADET, IOWA STATE UNIVERSITY, C. MISCHKE
      XL=XA
      XR=XB
      Q=10.E-07
      IF(F.LT.-Q) GO TO 41
      IF(F.GT.Q) GO TO 42
      IF(F.GT.-Q.AND.F.LT.Q) GO TO 43
   41 ICODE=-1
      GO TO 100
   42 ICODE=1
      GO TO 100
   43 ICODE=0
      F=ICODE
      GO TO 100
  111 IF(K)32,31,32
   32 WRITE(6,33)
   33 FORMAT(' CONVERGENCE MONITOR IOWA CADET SUBROUTINE GOLD',/,
     1' VERSION 11/76 C. MISCHKE',/,/,
     2'      N          Y1              Y2              X1              X2'
     3,/,/)
   31 N=0
      XLEFT=XL
      XRIGHT=XR
   13 SPAN=XR-XL
      DELTA=ABS(SPAN)
   14 X1=XL+0.381966*DELTA
      X2=XL+0.618034*DELTA
      CALL MERIT1(X1,Y1)
      CALL MERIT1(X2,Y2)
      N=N+2
    3 IF(K)34,9,34
   34 WRITE(6,35)N,Y1,Y2,X1,X2
   35 FORMAT(I5,4(1X,G15.7))
    9 IF(ICODE)50,50,51
   50 IF(0.381966*DELTA-ABS(F))4,4,8
   51 IF(0.618034*(XR-XL)-F*SPAN)4,4,8
    8 DELTA=0.618034*DELTA
      IF(Y1-Y2)1,10,2
    1 XL=X1
      X1=X2
      Y1=Y2
      X2=XL+0.618034*DELTA
      CALL MERIT1(X2,Y2)
      N=N+1
      GO TO 3
```

```
  2 XR=X2
    Y2=Y1
    X2=X1
    X1=XL+0.381966*DELTA
    CALL MERIT1(X1,Y1)
    N=N+1
    GO TO 3
  4 IF(Y2-Y1)5,5,6
  5 YBIG=Y1
    XBIG=X1
    XL1=XL
    XR1=X2
    GO TO 39
  6 YBIG=Y2
    XBIG=X2
    XL1=X1
    XR1=XR
    GO TO 39
 10 XL=X1
    XR=X2
    DELTA=XR-XL
    GO TO 14
 39 IF(K)40,40,37
 37 IF(ICODE)60,60,61
 60 A=-F
    WRITE(6,138)A
138 FORMAT(/,/,
   1' ACCEPTABLE BILATERAL TOLERANCE ON XSTAR ..............',G15.7)
    GO TO 140
 61 WRITE(6,139)F
139  FORMAT(/,/,
   1' FRACTIONAL REDUCTION IN INTERVAL OF UNCERTAINTY ......',G15.7)
140 WRITE(6,38)XLEFT,XRIGHT,YBIG,XBIG,XL1,XR1,N
 38 FORMAT(/,
   1' LEFTHAND ABSCISSA OF INTERVAL OF UNCERTAINTY .........',G15.7,/,
   2' RIGHTHAND ABSCISSA OF INTERVAL OF UNCERTAINTY .........',G15.7,/,
   3' EXTREME ORDINATE DISCOVERED DURING SEARCH .............',G15.7,/,
   4' ABSCISSA OF EXTREME ORDINATE ..........................',G15.7,/,
   5' NEW LEFTHAND ABSCISSA OF INTERVAL OF UNCERTAINTY .....',G15.7,/,
   6' NEW RIGHTHAND ABSCISSA OF INTERVAL OF UNCERTAINTY ....',G15.7,/,
   7' NUMBER OF FUNCTION EVALUATIONS EXPENDED DURING SEARCH ',I11,/,/)
 40 XL=XLEFT
    XR=XRIGHT
112 RETURN
100 IERROR=0
    IF(K)102,101,101
101  IF(K-1)104,104,102
102 WRITE(6,103)K
103 FORMAT(' *****ERROR MESSAGE SUBROUTINE GOLD*****',/,
   1'       I1,',I15,' IS NOT 0 OR 1')
    IERROR=IERROR+1
104 IF(XR-XL)105,105,1070
105 WRITE(6,106)XL,XR
106 FORMAT(' *****ERROR MESSAGE SUBROUTINE GOLD*****',/,
   1'       A2,',G15.7,' CANNOT BE .GE. A3,',G15.7)
    IERROR=IERROR+1
1070 IF(ICODE.NE.0) GO TO 107
 120 WRITE(6,121)F
 121 FORMAT(' *****ERROR MESSAGE SUBROUTINE GOLD*****',/,
   1'       VALUE OF A4,',G15.7,' CANNOT BE ZERO')
    IERROR=IERROR+1
 107 IF(ICODE)113,115,1107
1107 IF(F.GT.0..AND.F.LT.1.) GO TO 115
    WRITE(6,110)F
 110 FORMAT(' *****ERROR MESSAGE SUBROUTINE GOLD*****',/,
   1'       A4,'G15.7,' DOES NOT LIE BETWEEN 0. AND 1.')
    IERROR=IERROR+1
    GO TO 115
 113 IF(ABS(F).LT.ABS(XR-XL)/2.) GO TO 115
    WRITE(6,114)F
 114 FORMAT(' *****ERROR MESSAGE SUBROUTINE GOLD*****',/,
   1'       ABSOLUTE VALUE OF A4,',G15.7, ' .GE.(A2 MINUS A3)/2')
    IERROR=IERROR+1
 115 IF(IERROR)111,111,112
    END
```

One-Dimensional Golden Section Search
GOLD(I1, A2, A3, A4, A5, B1, B2, B3, B4, J5) Mischke

This subroutine will search over a one-dimensional unimodal function and report the largest ordinate found, its abscissa, final abscissas bound in the interval of uncertainty, and the number of function evaluations expended during the search.

The subroutine requires the specification of the present interval of uncertainty, the fractional reduction required in the interval of uncertainty (or bilateral tolerance on abscissa of the extreme), and whether or not a convergence monitor printout is desired. The necessary number of function evaluations may be predicted from

$$N = \left[1 + 2.08 \ln \frac{1}{F}\right]_+$$

when the fractional reduction in interval of uncertainty F is given, or

$$N = \left[2.08 \ln \frac{x_r - x_\ell}{a}\right]_+$$

when the bilateral tolerance a on abscissa of extreme is given.

See *Introduction to Computer-Aided Design*, C. Mischke, Prentice-Hall, 1968, p. 64, or *Mathematical Model Building*, 2nd rev. ed., C. Mischke, Iowa State University Press, 1980, pp. 282–290.

CALLING PROGRAM REQUIREMENTS

Provide a subroutine A5(X,Y) which returns the ordinate Y when the abscissa X is tendered.

Provide the equivalent of the following statement: EXTERNAL A5

CALL LIST ARGUMENTS:

I1 = 0,	convergence monitor will not print
= 1	convergence monitor will print
A2 = x_ℓ	original left-hand abscissa of interval of uncertainty
A3 = x_r	original right hand abscissa of interval of uncertainty
A4 =	fractional reduction in interval of uncertainty desired, F, entered positive or bilateral tolerance on abscissa at extreme, a, entered negative
A5 =	name of the one-dimensional unimodal function SUBROUTINE
B1 = y^*,	extreme ordinate discovered during search (maximum)
B2 = x^*,	abscissa of extreme ordinate
B3 = x_1,	final left-hand abscissa of interval of uncertainty
B4 = x_2,	final right-hand abscissa of interval of uncertainty
J5 = N,	number of function evaluations expended during search

PREEMPTED NAMES:

None

SIZE:

4264 bytes WATFIV compiler.

FIGURE 11.1 The documentation page of subroutine GOLD.

11.5 QUADRATURE

Another numerical chore is integration. Fortunately, Simpson's first rule is simple, robust, and surprisingly accurate. It is applied two panels at a time with equally spaced ordinates. A parabola is passed through the three ordinate points ([11.5], p. 79). If h is the ordinate spacing, then for the three abscissas x_0, x_1, and x_2,

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi) \tag{11.15}$$

where ξ is in the interval (x_0, x_2). The right-hand term is Richardson's error term, which is exact for some ξ which is generally unknown a priori. When this two-panel, three-ordinate operation is repeated a number of times in the interval a, b, then

$$\int_a^b f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{2n-1}) + f(x_{2n})] - \frac{h^5}{90}\sum_{i=1}^{n} f^{(4)}(\xi_i) \tag{11.16}$$

where n is the number of applications of the two-panel ritual, $2n$ is the number of panels, and $2n + 1$ is the number of function evaluations. There is great merit in making the number of panels an even number divisible by 4. The ordinate spacing h is given by $h = (b - a)/(2n)$, and so the error term becomes, removing the summation sign,

$$E_n \doteq \frac{(b-a)^5}{2880n^4} f^{(4)}(\xi) \tag{11.17}$$

By evaluating the integral using n_2 applications, then again with n_1 applications, from Eq. (11.17),

$$\frac{E_{n_2}}{E_{n_1}} = \left(\frac{n_2}{n_1}\right)^4 \tag{11.18}$$

from which

$$n_2 = n_1 \left|\frac{E_{n_1}}{E_{n_2}}\right|^{1/4} = n_1 \left|\frac{E_{n_1}}{\Delta}\right|^{1/4} \tag{11.19}$$

where Δ is the tolerable error. If n_1 is the number of applications of the rule in the interval a, b and n_2 is the number of applications in another evaluation in the same interval a, b, then the value of the integral I is

$$I \doteq I_{n_1} + E_{n_1} \doteq I_{n_2} + E_{n_2}$$

If n_2 is one-half of n_1, then combining Eq. (11.18) with the above equation results in

$$I \doteq I_{n_1} + \frac{I_{n_1} - I_{n_2}}{15} \tag{11.20}$$

Example 1. Evaluate the integral $\int_1^2 dx/x$.

(*a*) Using two applications of Simpson's rule, estimate the error in $I_{n=2}$. (*b*) For an error of the magnitude 0.000 01, estimate the number of applications necessary, and (*c*) integrate and examine the error.

Solution. (*a*) Using two applications of Simpson's rule,

x	$1/x$	Mult.	
1.00	1.000 000 000	1	1.000 000 000
1.25	0.800 000 000	4	3.200 000 000
1.50	0.666 666 667	2	1.333 333 333
1.75	0.571 428 571	4	2.285 714 286
2.00	0.500 000 000	1	0.500 000 000
		$\Sigma =$	8.319 047 619

From the first part of Eq. (11.16),

$$I_{n=2} = \frac{0.25}{3}(8.319\,047\,619) = 0.693\,253\,968$$

For one application of Simpson's rule,

x	$1/x$	Mult.	
1.00	1.000 000 000	1	1.000 000 000
1.50	0.666 666 667	4	2.666 666 667
2.00	0.500 000 000	1	0.500 000 000
		$\Sigma =$	4.166 666 667

$$I_{n=1} = \frac{0.5}{3}(4.166\,666\,667) = 0.694\,444\,445$$

From the second part of Eq. (11.20),

$$E_{n=2} = \frac{I_{n=2} - I_{n=1}}{15} = \frac{0.693\,253\,968 - 0.694\,444\,445}{15}$$

$$= -0.000\,079\,565$$

(*b*) From Eq. (11.19),

$$n_2 = n_1 \left|\frac{E_{n_1}}{E_{n_2}}\right|^{1/4} = 2\left|\frac{-0.000\,079\,365}{0.000\,01}\right|^{1/4} = 3.36 \Rightarrow 4$$

(*c*) Using four applications,

x	$1/x$	Mult.	
1.000	1.000 000 000	1	1.000 000 000
1.125	0.888 888 889	4	3.555 555 556
1.250	0.800 000 000	2	1.600 000 000
1.375	0.727 272 727	4	2.909 090 903
1.500	0.666 666 667	2	1.333 333 333
1.625	0.615 384 615	4	2.461 538 462
1.750	0.571 428 571	2	1.142 857 143
1.875	0.533 333 333	4	2.133 333 333
2.000	0.500 000 000	1	0.500 000 000
		$\Sigma =$	16.635 708 73

$$I_{n=4} = \frac{0.125}{3}(16.635\,708\,73) = 0.693\,154\,530$$

The true value of the integral is ln 2 = 0.693 147 181. The value of $I_{n=4}$ differs by 1 in the fifth decimal place. Furthermore, we can improve $I_{n=4}$ using Eq. (11.20):

$$I \doteq I_{n=4} + E_{n=4} \doteq I_{n=4} + \frac{I_{n=4} - I_{n=2}}{15}$$

$$= 0.693\,154\,530 - 0.000\,006\,629 = 0.693\,147\,901$$

and we have six correct digits, a bonus since we estimated $E_{n=4}$ along the way.

The use of Simpson's rule to evaluate an integral is both controllably accurate and relatively simple.

11.6 CHECKING

It is useful to check intermediate results on an as-you-go basis as well as upon completion. If there are one hundred subtasks involved in completing an engineering task, ponder this: If your average reliability in performing each subtask correctly is 0.99, then the probability of performing them in sequence correctly is 0.99^{100}, or 0.37. How does one improve such a performance? One way is by checking. If the hundred steps are concatenated to reach the result, where in the chain of events is a mistake most wasteful in effort because work has to be repeated? Early! When do most people think of checking? At the end. Checking steps as they are done, checking groups of steps, and checking the final result is an appropriate and wise course of action.

11.6.1 Limiting-Case Check

This method of checking a derived equation allows the parameters, in turn, to range from the point of vanishing to increasing without upper bound. Do the results still make sense? Do the results contract to a previously known correct result? Making sense is a necessary but not sufficient condition.

11.6.2 Dimensional Check

Equations should be dimensionally homogeneous. Apply the dimensional operator dim( ) to every term in an equation, substituting the fundamental dimensions term by term. Remember that the result of applying the operator to a dimensionless term is unity. Dimensional homogeneity is a necessary but not sufficient condition.

11.6.3 Experience

Our lifetime experience with similar things will suggest "expected relationships": symmetries of certain forms, indirect and direct proportionalities, and nonlinearities of a particular order, such as proportionality to the cube of some parameter. All these little tidbits of reality from prior contexts can be examined for applica-

bility to the case at hand. Congruence with experience is a necessary but not sufficient condition.

11.6.4 Robustness of Assumptions

Deductions from first principles and cause-effect-extent mathematical models depend on assumptions such as "friction is negligible" or "radiation is secondary." What we are really saying is that the result will be useful to our purpose—that is, *robust*—even if the influences of friction or radiation are ignored. The mathematical meaning of the word *assumption* does not fully apply here, nor does it serve us well. In reality, we have made a *decision* based on an experiential value judgment that acting on the result is prudent and resources are risked at a very small, acceptable level. Engineers should treat all such "assumptions" as the decisions they really are. It is useful to ask

- Was it necessary to make this decision (assumption)?
- Has embracing it hidden an important influence of the surroundings on matter in the system or control region?
- Did I qualify my result with an explicit statement of this decision (assumption)?
- Is this decision (assumption) defendable at all values of the parameters that will be encountered?
- Did this decision (assumption) make the model sufficiently incongruent with nature to lose robustness?

Thoughtful responses to questions such as these can help uncover sketchy work. Engineers are responsible for all the decisions they make, whether by commission or by omission. It is prudent to list (call out) all such decisions (assumptions) in the design notebook, and the responses to queries such as those above in the check steps, so that the original engineer at a later date, or another engineer at any time, can understand, appreciate, and possibly challenge them.

11.6.5 Experiment

Results can be verified by experiment. In order to check a spring rate formulation, such as

$$k = \frac{d^4 G}{8D^3 N_a}$$

we express it in dimensionless form, Eq. (11.10):

$$\frac{k}{GD} = \frac{1}{8N_a}\left(\frac{d}{D}\right)^4 = \pi_1 = \frac{\pi_3{}^4}{8\pi_2}$$

and check the two nuggets of reality, $\pi_1 \pi_2 = C_1$ and $\pi_1/\pi_3^4 = C_2$, which were the bases for the evaluation of the partial derivatives of Eq. (11.9). The first lends itself to a linear plot of the form $\pi_1 = C_1/\pi_2$. Ideally this should lead to a straight data string on a plot of π_1 versus $1/\pi_2$ which lines up with the origin. If one has several springs which

differ in turns count only, placing known weights on the spring and measuring the deflection with a dial indicator can supply some data points.

If a least-squares fit of the form $\pi_1 = a + b/\pi_2$ misses the origin, has the origin really been missed? There are statistical methods for saying that with the data you have, the origin has been hit (statistically) or missed (statistically) and quantifying the level of risk in believing either. For example, the number of dead turns N_d comes from the equation $N_t = N_d + N_a$. The number of dead turns N_d may not be precisely 2, depending on how the squared and ground end turns are actually formed. The determination of N_a appears to be a counting procedure; that is, $N_a = N_t - N_d = N_t - 2$ implies great precision, except that the number 2 can be suspect.

The second experimental check on $\pi_1 = C_2\pi_3^4$ can be done on a log-log plot.

The final form constant C in Eq. (11.10) can be found from the experimental data, and again, statistical methods will develop the chances that C is 1/8.

Notice that the economy of effort of Sec. 11.2 is used to make an experimental check one of least cost. An equation that is an old familiar friend, when applied outside its domain of validity, can't play the game well, or at all. The experimental check can detect this.

11.6.6 Alternative Method of Derivation

If one is truly at "the cutting edge," one rejoices at achieving a result, and urging a second approach, say an energy method, may not be helpful. Nevertheless, we are rarely at the edge, and an alternative approach is a possible and useful method of check. For example, an analog equation can be found.

11.6.7 Have a Colleague Check Your Work

Often, a colleague was educated at a different school by a different faculty using differing emphases and methodologies. Some of these may not be familiar to you, or of first choice. Having a colleague check brings not only a fresh viewpoint but a different ensemble of experience to the problem. While a challenge to some of your decisions may result, it should be welcomed in the pursuit of soundness and completeness of analysis and documentation.

11.6.8 The Insufficiency of Checking Methods

Methods of checking are directed toward verification of matters of mathematical necessity but not sufficiency. Additionally, the limiting-case check, dimensional check, experience check, and assumptions check will not uncover an error such as in the 8 of the spring example. The experimental check *can,* the alternative method check *may,* and the colleague check *might* detect it.

Methods of checking are ways of detecting troubles. In themselves they do not rectify troubles. Being unable to assure infallibility, engineers check, check, and check again.

11.6.9 Checking the Problem-Solving Strategy

Failure to achieve a solution or ineffective progress in the pursuit of a solution can be traceable to problem-solving methodology. Previously identified checks were

focused on technical matters, usually mathematical modeling. Problem-solving strategies are more global, more qualitative, and less tangible. The following ideas and questions can be useful in encouraging a healthy skepticism.

There are three clearly identifiable steps in problem solving: (1) defining the problem, (2) planning its treatment, and (3) executing the plan. There are two more steps which are not sequential, but are woven into defining, planning, and executing as necessary. They are also the final two steps following the completion of the execution step. These are (4) checking and (5) learning and generalizing. In more detail, the steps consist of asking the following questions:

Defining the Problem

- What is the real problem or issue?
- What questions are to be answered?
- What are the pertinent facts?
- If several problems are present, which should be addressed first?

Planning Its Treatment

- How can I solve the problem?
- What fundamental principles apply?
- What general truths will help toward a solution?
- What is my plan to move from what is known to what I want?
- Is my plan sufficiently complete for execution? Has any other work been done on this problem?

Executing the Plan

- What is the result of my plan?
- How do I get a useful result from the principle applied?
- Where am I with respect to my plan?

Checking

- Is my work correct in every detail?
- Are assumptions (decisions) reasonable?
- Have I considered *all* important factors?
- Do the results make good sense?
- Have I applied all methods of checking?

Learning and Generalizing

- What have I found out?
- What does the result tell me about the answer to the original problem?
- What does the result mean, and what is its interpretation in common terms?

- How may my results have been affected by my assumptions (decisions)?
- Is the result good enough to act upon, or must the solution be refined?

In moments of doubt as to what to do next, ask

- What do I really want to know?
- What am I doing now?
- Why?
- Will it help?

All this is a demonstration of the sagacity of the adage, "There are no right answers, only right questions."

11.6.10 Checking Cause-Effect-Extent Models

Engineers tend to be self-sufficient in mastering cause-effect-extent models of system/surroundings interactions. This in turn leads them to rarely check to see if some relevant caveat has been ignored. For example, changes in system *internal* energy are those that occur in the absence of gravitation, motion, charge, magnetism, and capillarity. If internal energy is a consideration, one should check to be sure that these things are absent, inconsequential in magnitude, or accounted for in some other way. Since this kind of information is scattered in many books on various subjects, it can be helpful to consult Ref. [11.6], which treats more than a hundred effects by providing descriptions, illustrations, magnitude relations, and references.

11.6.11 Checking Personal Competence

There are times when every engineer is "in over his or her head" and outside his or her personal knowledge and experience base. This happens occasionally because no one can predict where a solution will lead. Engineers do not like to talk about this. The best remedy is knowing when to seek help, or the resources to acquire that help.

11.6.12 The Final Adequacy Assessment as a Final Check

An adequacy assessment (Sec. 5.2) consists of those cerebral and empirical steps that convince the designer that the specification set represents a robust design. The recommended step before "turning work in" is a final adequacy assessment. It should begin not with what is in your head, but with information taken directly from the report and drawings that will leave your desk. Engineers think in terms of significant attributes (a midrange length, a smallest diameter, etc.). These should not be remembered, but reconstructed from your specifications. If you have been thinking in terms of, say, active spring turns, and you have entered that in the spring maker's form blank for total turns, you will set in motion mass production of springs that are not what you had in mind. By starting the final adequacy assessment check with what leaves your desk, you can catch these kinds of errors.

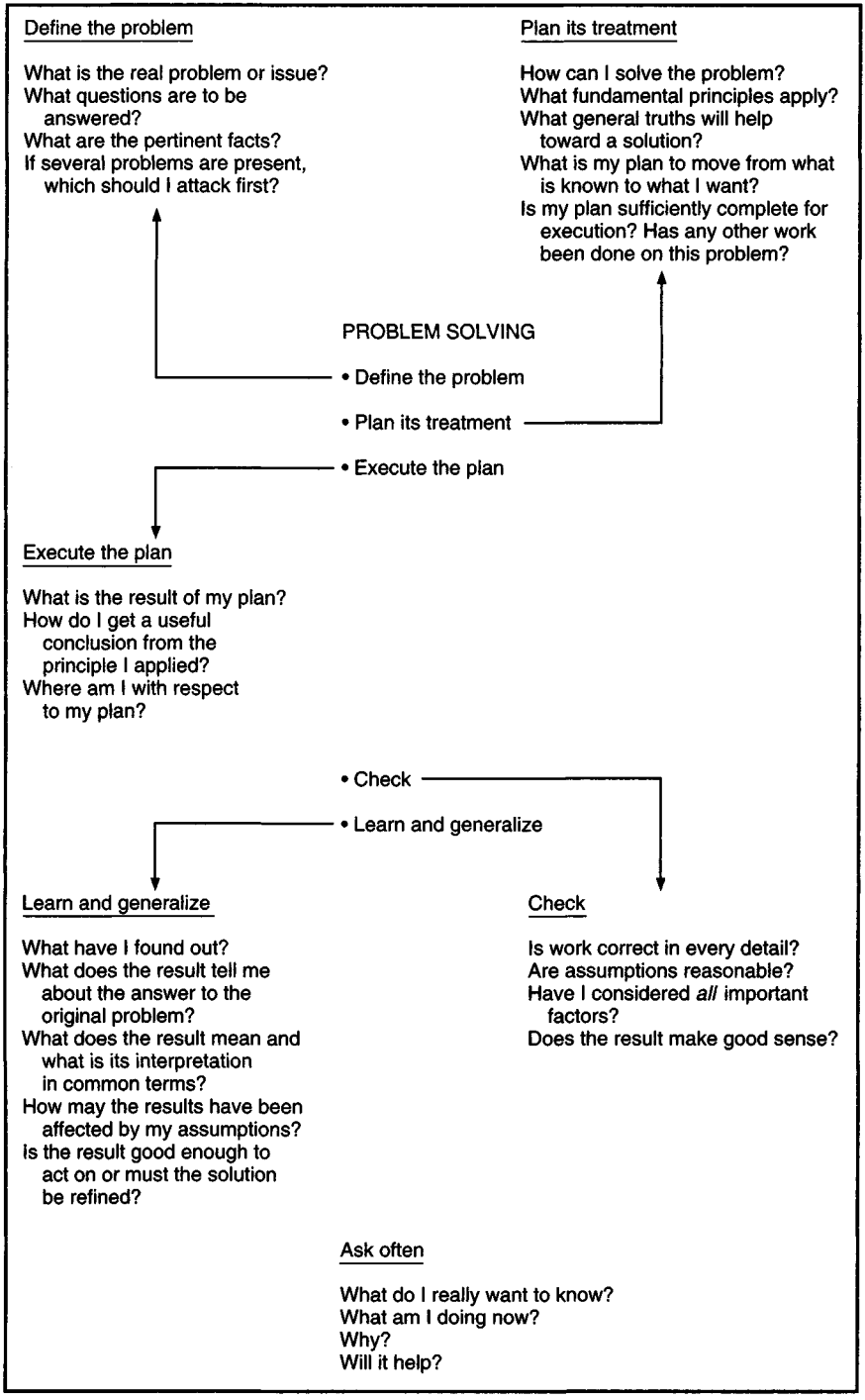

FIGURE 11.2 Some pertinent questions to ask oneself while solving problems. The check, learn and generalize steps are to be woven into the first three, and are the last ones.

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